ski

Bertrand Simon

part of a joint work with:
Bender, Berry, Johnson, Kroeger, McCauley, Phillips, Singh, Zage

ENS Lyon

Jan. 2018

Cache-efficient skip lists

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Outline

The problem we want to solve

Dictionary problem on $\mathbb{N}$

- Insert i
- Delete i
- Search i
- Range Query (i, k elements)

Example

Insert 26; Insert 8; Insert 4;
Insert 17; Insert 42; Insert 1664;
Delete 4; Search 26; Delete 26;
Insert 58; Insert 2; Search 26;
$RQ(8,4) \rightarrow [8;17;42;58]$;

Performance we seek (n elements in the set)

- Insert, Delete, Search:
- Range Query:

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Famous data structures solve this

- Self-balancing binary search trees (AVL, Red-black tree...)

What's the use of skip lists?

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  Improved in 1993, 1999, 2001, 2008, 2011
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More

- Easy concurrency
- fun, elegant, teaches probabilities...

From a simple list to skip lists

Properties

- Maintain a sorted list of the elements
- Support operations in $\mathcal{O}(\log n)$ in expectation and with high probability ($\approx$ worst-case analysis)

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- $\forall c$ large, with proba $1 - n^{-\Omega(c)}$, all operations cost $< c \log n$
- Ex: $n = 1000$, $1 - 10^{-9}$ $< 3 \log n$

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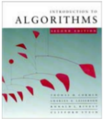
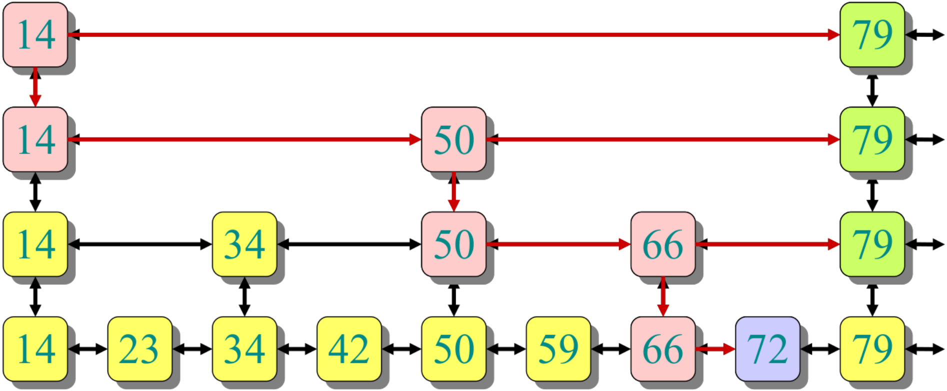
Description of *ideal* skip lists without updates

On the board



Searching in lg *n* linked lists

EXAMPLE: SEARCH(72)



Updating a skip list

Updating ideal skip lists: expensive

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Do you see something missing?

Some probabilities

Theorem

A skip list has $\mathcal{O}(\log n)$ levels whp.

Proof.

$$\begin{aligned}\mathcal{P}(> c \log n \text{ levels}) &\leq n \cdot \mathcal{P}(\text{Insert gets} > c \log n \text{ promotions}) \\ &\leq n \cdot \left(\frac{1}{2}\right)^{c \log n} \\ &\leq n^{1-c}\end{aligned}$$

□

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- if the node was promoted: go up (proba. $1/2$)
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- we stop after $< c \log n$ "up" moves

Whp, after how many moves do we stop?

Answer:

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Forget everything you know

Classic RAM model used to evaluate algorithm

- Memory access (read, write)
- Computation (compare, add, multiply. . .)

} cost 1

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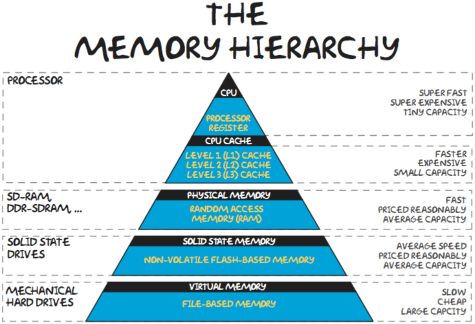
Problem when dealing with large data



A new model

Change of view

- *Classic* complexity (RAM model): focus on computations
- Disk-Access Model [Aggarwal'88] : focus on communications

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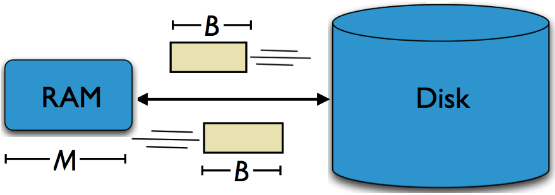
Model

- Two layers of memory: a main RAM of size M and an infinite disk
- Data needs to be on RAM to be processed
- Can exchange contiguous blocks of size B for 1 I/O



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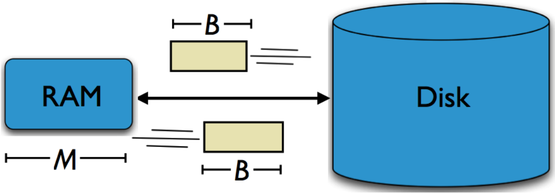
- Complexity of an algorithm: worst-case I/O number



Why are I/Os so important?

Large data: classic algorithms access frequently to disk

Access time

- RAM: 100 ns
- Disk: 10 ms = 10 000 000 ns

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Access time

- RAM: 100 ns
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- Analogy: $\frac{\text{Ram speed}}{\text{Disk speed}} \approx \frac{\text{escape velocity from Earth}}{\text{speed of a turtle}}$

DAM model: totally forget computations

New bounds

Classic bounds

	RAM	DAM (I/Os)
Scan	N	
Search	$\log N$	
Merge-Sort	$N \log N$	

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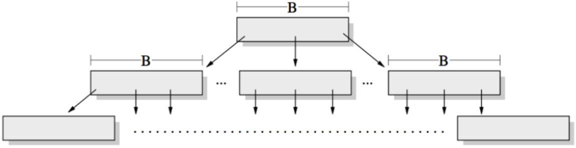
External memory Search tree: B-tree



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Merge-Sort	$N \log N$	$\frac{N}{B} \log_{M/B} \frac{N}{B}$

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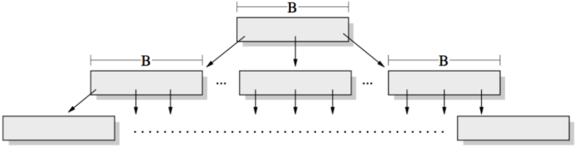

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Why it does not work straight away

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Any idea to improve locality? *(& keep history-independence)*

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- Block together elements between 2 promoted ones
- Change the promotion probability

What should be the promotion probability?

If $p > 1/B$

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If $p = 1/B$ **[Golovin'2010]**

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If $p < 1/B$

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If $p = 1/B$ **[Golovin'2010]**

- OK on average
- Whp: $\sqrt{N}$ series of $B \log N$ non-promoted elements
- For $> \sqrt{N}$ elements, a search costs $\Omega(\log N)$ I/Os

Towards our skip list

Promotion probability

- $\frac{\log B}{B} < p < B^{-0.5}$ (ex: $p = B^{-0.7}$) $\longrightarrow$ searches OK on average
- largest series: $< B \log_B N$ whp $\longrightarrow$ $O(\log_B N)$ I/Os for searches

Blocking strategy

- Block between doubly-promoted elements $\longrightarrow$ Range Queries
- Reserve buffers between promoted elements $\longrightarrow$ Updates

More

- Some tricks to ensure all bounds whp & history independence

Example of our skip list for $B = 3$ and $p = 1/2$